

THE REVERSE ORDER LAW FOR MOORE-PENROSE INVERSES OF OPERATORS ON HILBERT C*-MODULES

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ABSTRACT. Suppose T and S are bounded adjointable operators between Hilbert C*-modules admitting bounded Moore-Penrose inverse operators. Some necessary and sufficient conditions are given for the reverse order law $(TS)^\dagger = S^\dagger T^\dagger$ to hold. In particular, we show that the equality holds if and only if $\text{Ran}(T^*TS) \subseteq \text{Ran}(S)$ and $\text{Ran}(SS^*T^*) \subseteq \text{Ran}(T^*)$, which was studied first by Greville [*SIAM Rev.* 8 (1966) 518–521] for matrices.

1. INTRODUCTION AND PRELIMINARIES.

It is well-known that for invertible operators (or nonsingular matrices) T, S and TS , $(TS)^{-1} = S^{-1}T^{-1}$. However, this so-called reverse order law is not necessarily true for other kind of generalized inverses. An interesting problem is, for given operators (or matrices) TS with TS meaningful, then under what conditions, $(TS)^\dagger = S^\dagger T^\dagger$? The problem first studied by Greville [7] and then reconsidered by Bouldin and Izumino [2, 9]. Many authors discussed the problem like this, see e.g. [3, 4, 5, 11, 13] and references therein. An special case, when $S = T^*$, was given by Moslehian et al. [14] for a Moore-Penrose invertible operator T on Hilbert C*-modules. The later paper and the work of [5, 7] motivate us to study the problem in the framework of Hilbert C*-modules.

The notion of a Hilbert C*-module is a generalization of the notion of a Hilbert space. However, some well known properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold in the framework Hilbert modules. The first use of such objects was made by I. Kaplansky [10] and then studied more in the work of W. L. Paschke [15]. Let us quickly recall the definition of a Hilbert C*-module.

Suppose that \mathcal{A} is an arbitrary C*-algebra and E is a linear space which is a right \mathcal{A} -module and the scalar multiplication satisfies $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in E$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. The \mathcal{A} -module E is called a *pre-Hilbert \mathcal{A} -module* if there exists an \mathcal{A} -valued map $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathcal{A}$ with the following properties:

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- (i) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$; for all $x, y, z \in E, \lambda \in \mathbb{C}$,
- (ii) $\langle x, ya \rangle = \langle x, y \rangle a$; for all $x, y \in E$ and $a \in A$,
- (iii) $\langle x, y \rangle^* = \langle y, x \rangle$; for all $x, y \in E$,
- (iv) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

The \mathcal{A} -module E is called a *Hilbert C^* -module* if E is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. For any pair of Hilbert C^* -modules E_1 and E_2 , we define $E_1 \oplus E_2 = \{(e_1, e_2) \mid e_1 \in E_1 \text{ and } e_2 \in E_2\}$ which is also a Hilbert C^* -module whose \mathcal{A} -valued inner product is given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \text{ for } x_1, x_2 \in E_1 \text{ and } y_1, y_2 \in E_2.$$

If V is a (possibly non-closed) \mathcal{A} -submodule of E , then $V^\perp := \{y \in E : \langle x, y \rangle = 0, \text{ for all } x \in V\}$ is a closed \mathcal{A} -submodule of E and $\overline{V} \subseteq V^{\perp\perp}$. A Hilbert \mathcal{A} -submodule V of a Hilbert \mathcal{A} -module E is orthogonally complemented if V and its orthogonal complement V^\perp yield $E = V \oplus V^\perp$, in this case, V and its biorthogonal complement $V^{\perp\perp}$ coincide. For the basic theory of Hilbert C^* -modules we refer to the book by E. C. Lance [12]. Note that every Hilbert space is a Hilbert \mathbb{C} -module and every C^* -algebra \mathcal{A} can be regarded as a Hilbert \mathcal{A} -module via $\langle a, b \rangle = a^*b$ when $a, b \in \mathcal{A}$.

Throughout this paper we assume that \mathcal{A} is an arbitrary C^* -algebra. We use $[\cdot, \cdot]$ for commutator of two elements. The notations $Ker(\cdot)$ and $Ran(\cdot)$ stand for kernel and range of operators, respectively. Suppose E and F are Hilbert \mathcal{A} -modules, $\mathcal{L}(E, F)$ denotes the set of all bounded adjointable operators from E to F , that is, all operator $T : E \rightarrow F$ for which there exists $T^* : F \rightarrow E$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$, for all $x \in E$ and $y \in F$.

Closed submodules of Hilbert modules need not to be orthogonally complemented at all, however we have the following well known results. Suppose T in $\mathcal{L}(E, F)$, the operator T has closed range if and only if T^* has. In this case, $E = Ker(T) \oplus Ran(T^*)$ and $F = Ker(T^*) \oplus Ran(T)$, cf. [12, Theorem 3.2]. In view of [16, Lemma 2.1], $Ran(T)$ is closed if and only if $Ran(TT^*)$ is, and in this case, $Ran(T) = Ran(TT^*)$.

Let $T \in \mathcal{L}(E, F)$. The Moore-Penrose inverse T^\dagger of T (if it exists) is an element $X \in \mathcal{L}(F, E)$ which satisfies

- (1) $TXT = T$,
- (2) $XTX = X$,
- (3) $(TX)^* = TX$,
- (4) $(XT)^* = XT$.

If $\theta \subseteq \{1, 2, 3, 4\}$, and X satisfies the equations (i) for all $i \in \theta$, then X is an θ -inverse of T . The set of all θ -inverses of T is denoted by $T\{\theta\}$. In particular, $T\{1, 2, 3, 4\} = \{T^\dagger\}$. The properties (1) to (4) imply that T^\dagger is unique and $T^\dagger T$ and $T T^\dagger$ are orthogonal projections. Moreover, $\text{Ran}(T^\dagger) = \text{Ran}(T^\dagger T)$, $\text{Ran}(T) = \text{Ran}(T T^\dagger)$, $\text{Ker}(T) = \text{Ker}(T^\dagger T)$ and $\text{Ker}(T^\dagger) = \text{Ker}(T T^\dagger)$ which lead us to $E = \text{Ker}(T^\dagger T) \oplus \text{Ran}(T^\dagger T) = \text{Ker}(T) \oplus \text{Ran}(T^\dagger)$ and $F = \text{Ker}(T^\dagger) \oplus \text{Ran}(T)$. We also have $\text{Ran}(T^\dagger) = \text{Ran}(T^*)$ and $\text{Ker}(T^\dagger) = \text{Ker}(T^*)$.

Xu and Sheng in [19] have shown that a bounded adjointable operator between two Hilbert C^* -modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its Moore-Penrose, see [6, 8, 16, 18] for more detailed information.

It is a classical result of Greville [7], that $(TS)^\dagger = S^\dagger T^\dagger$ if and only if $T^\dagger T S S^* T^* = S S^* T^*$ and $S S^\dagger T^* T S = T^* T S$ (or equivalently, $\text{Ran}(S S^* T^*) \subseteq \text{Ran}(T^*)$ and $\text{Ran}(T^* T S) \subseteq \text{Ran}(S)$) for Moore-Penrose invertible matrices T and S . The present paper is an extension of some results of [5, 7, 14] to Hilbert C^* -modules settings. Indeed, we give some necessary and sufficient conditions for reverse order law for the Moore-Penrose inverse by using the matrix form of bounded adjointable module maps. These enable us to derive Greville's result for bounded adjointable module maps.

The matrix form of a bounded adjointable operator $T \in \mathcal{L}(E, F)$ is induced by some natural decompositions of Hilbert C^* -modules. If $F = M \oplus M^\perp, E = K \oplus K^\perp$ then T can be written as the following 2×2 matrix

$$(1.1) \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

with operator entries, $T_1 \in \mathcal{L}(K, M), T_2 \in \mathcal{L}(K^\perp, M), T_3 \in \mathcal{L}(K, M^\perp)$ and $T_4 \in \mathcal{L}(K^\perp, M^\perp)$.

Lemma 1.1. *Let $T \in \mathcal{L}(E, F)$ have a closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of submodules $E = \text{Ran}(T^*) \oplus \text{Ker}(T)$ and $F = \text{Ran}(T) \oplus \text{Ker}(T^*)$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix}.$$

Proof. The operator T and its adjoint T^* have the following representations:

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix},$$

$$T^* = \begin{bmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix}.$$

From $T^*(\text{Ker}(T^*)) = \{0\}$ we obtain $T_3^* = 0$ and $T_4^* = 0$, so $T_3 = 0$ and $T_4 = 0$. Since $T(\text{Ker}(T)) = \{0\}$, $T_2 = 0$ and so $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}$.

Since $\text{Ran}(T)$ is close, T_1 possesses a bounded adjointable inverse from $\text{Ran}(T)$ onto $\text{Ran}(T^*)$. Now, it is easy to check that the matrix $\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is the Moore–Penrose inverse of $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}$. \square

Lemma 1.2. *let $T \in \mathcal{L}(E, F)$ have a closed range. Let E_1, E_2 be closed submodules of E and F_1, F_2 be closed submodules of F such that $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of submodules $E = \text{Ran}(T^*) \oplus \text{Ker}(T)$ and $F = \text{Ran}(T) \oplus \text{Ker}(T^*)$:*

$$(1.2) \quad T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^* \in \mathcal{L}(\text{Ran}(T))$ is positive and invertible. Moreover,

$$(1.3) \quad T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix}.$$

$$(1.4) \quad T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

where $\mathfrak{D} = T_1^* T_1 + T_2^* T_2 \in \mathcal{L}(\text{Ran}(T^*))$ is positive and invertible. Moreover,

$$(1.5) \quad T^\dagger = \begin{bmatrix} \mathfrak{D}^{-1} T_1^* & \mathfrak{D}^{-1} T_2^* \\ 0 & 0 \end{bmatrix}.$$

Proof. We prove only the matrix representations (1.2) and (1.3), the proof of (1.4) and (1.5) are analogous. The operator T has the following representation:

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix},$$

which yields

$$T^* = \begin{bmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

From $T^*(\text{Ker}(T^*)) = \{0\}$ we obtain $T_3^* = 0$ and $T_4^* = 0$. Then $T_3 = 0$ and $T_4 = 0$ which yield the matrix form (1.2) of T . Consequently, the adjoint operator T^* has the matrix representation

$$T^* = \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

We therefore have

$$(1.6) \quad TT^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix}.$$

where $D = T_1T_1^* + T_2T_2^* : \text{Ran}(T) \rightarrow \text{Ran}(T)$. From $\text{Ker}(TT^*) = \text{Ker}(T^*)$ it follows that D is injective. From $\text{Ran}(TT^*) = \text{Ran}(T)$ it follows that D is surjective. Hence, D is invertible. Using [14, Corollary 2.4] and (1.6) we obtain

$$T^\dagger = T^*(TT^*)^\dagger = \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix}.$$

□

2. THE REVERSE ORDER LAW

We begin our section with the following useful facts about the product of module maps with closed range. Suppose E, F and G are Hilbert C^* -modules and $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(F, G)$ are bounded adjointable operators with closed ranges. Then TS has closed range if and only if $T^\dagger TSS^\dagger$ has, if and only if $\text{Ker}(T) + \text{Ran}(S)$ is an orthogonal summand in F , if and only if $\text{Ker}(S^*) + \text{Ran}(T^*)$ is an orthogonal summand in F . For the proofs of the results and historical notes about the problem we refer to [17] and references therein.

Theorem 2.1. *Suppose E, F and G are Hilbert C^* -modules and $S \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(F, G)$ and $TS \in \mathcal{L}(E, G)$ have closed ranges. Then following statements are equivalent:*

- (i) $TS(TS)^\dagger = TSS^\dagger T^\dagger$,
- (ii) $T^*TS = SS^\dagger T^*TS$,
- (iii) $S^\dagger T^\dagger \in (TS)\{1, 2, 3\}$.

Proof. Using Lemma 1.1, the operator S and its Moore-Penrose inverse S^\dagger have the following matrix forms:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(S^*) \\ \text{Ker}(S) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(S) \\ \text{Ker}(S^*) \end{bmatrix},$$

$$S^\dagger = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(S) \\ \text{Ker}(S^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(S^*) \\ \text{Ker}(S) \end{bmatrix}.$$

From Lemma 1.2 it follows that the operator T and T^\dagger have the following matrix forms:

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(S) \\ \text{Ker}(S^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T^*) \end{bmatrix},$$

$$T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^*$ is invertible and positive in $\mathcal{L}(\text{Ran}(T))$. Then we have the following products

$$TS = \begin{bmatrix} T_1 S_1 & 0 \\ 0 & 0 \end{bmatrix}, (TS)^\dagger = \begin{bmatrix} (T_1 S_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, S^\dagger T^\dagger = \begin{bmatrix} S_1^{-1} T_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that the following three expressions in terms of T_1 , T_2 and S_1 are equivalent to our statements.

- (1) $T_1 S_1 (T_1 S_1)^\dagger = T_1 T_1^* D^{-1}$, which is equivalent to (i).
- (2) $T_2^* T_1 = 0$, which is equivalent to (ii).
- (3) $T_1 T_1^* D^{-1} T_1 = T_1$ and $[T_1 T_1^*, D^{-1}] = 0$, which are equivalent to (iii).

Note that $[T_1 T_1^*, D^{-1}] = 0$, since $T_1 S_1 (T_1 S_1)^\dagger$ is selfadjoint. We show that (3) \Rightarrow (2) \Leftrightarrow (1) \Rightarrow (3).

To prove (1) \Leftrightarrow (2), we observe that $T_1 S_1 (T_1 S_1)^\dagger = T_1 T_1^* D^{-1}$ if and only if $(T_1 S_1)^\dagger = (T_1 S_1)^\dagger T_1 T_1^* D^{-1}$. The last statement is obtained by multiplying the first expression by $(T_1 S_1)^\dagger$ from the left side, or multiplying the second expression by $T_1 S_1$ from the left side, and using

$T_1 T_1^* = T_1 S_1 S_1^{-1} T_1^*$. We therefore have

$$\begin{aligned}
(T_1 S_1)^\dagger &= (T_1 S_1)^\dagger T_1 T_1^* D^{-1} \Leftrightarrow (T_1 S_1)^\dagger (T_1 T_1^* + T_2 T_2^*) = (T_1 S_1)^\dagger T_1 T_1^* \\
&\Leftrightarrow (T_1 S_1)^\dagger T_2 T_2^* = 0 \\
&\Leftrightarrow \text{Ran}(T_2 T_2^*) \subseteq \text{Ker}((T_1 S_1)^\dagger) = \text{Ker}((T_1 S_1)^*) \\
&\Leftrightarrow S_1^* T_1^* T_2 T_2^* = 0 \Leftrightarrow T_2 T_2^* T_1 = 0 \\
&\Leftrightarrow \text{Ran}(T_1) \subseteq \text{Ker}(T_2 T_2^*) = \text{Ker}(T_2^*) \\
&\Leftrightarrow T_2^* T_1 = 0.
\end{aligned}$$

To demonstrate (1) \Rightarrow (3), we multiply $T_1 S_1 (T_1 S_1)^\dagger = T_1 T_1^* D^{-1}$ by $T_1 S_1$ from the right side, we find $T_1 T_1^* D^{-1} T_1 = T_1$, i.e. (3) holds.

Finally, we prove (3) \Rightarrow (2). If $T_1 T_1^* D^{-1} T_1 = T_1$ and $[T_1 T_1^*, D^{-1}] = 0$, then $T_1 T_1^* T_1 = D T_1 = T_1 T_1^* T_1 + T_2 T_2^* T_1$. Consequently, $T_2 T_2^* T_1 = 0$ which implies $T_2 T_1^* = 0$, since $\text{Ran}(T_1) \subseteq \text{Ker}(T_2 T_2^*) = \text{Ker}(T_2^*)$. \square

Theorem 2.2. *Suppose E, F and G are Hilbert C^* -modules and $S \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(F, G)$ and $TS \in \mathcal{L}(E, G)$ have closed ranges. Then following statements are equivalent:*

- (i) $(TS)^\dagger TS = S^\dagger T^\dagger TS$,
- (ii) $TSS^* = TSS^* T^\dagger T$,
- (iii) $S^\dagger T^\dagger \in (TS)\{1, 2, 4\}$.

Proof. The operators T , S and TS and their Moore-Penrose inverses have the same matrix representations as in the previous theorem. To prove the assertions, we first find the equivalent expressions for our statements in terms of T_1 , T_2 and S_1 .

- (1) $(T_1 S_1)^\dagger T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1$, which is equivalent to (i).
- (2) $T_1 S_1 S_1^* T_1^* D^{-1} T_1 = T_1 S_1 S_1^*$ and $T_1 S_1 S_1^* T_1^* D^{-1} T_2 = 0$, which are equivalent to (ii).
- (3) $T_1 T_1^* D^{-1} T_1 = T_1$ and $[S_1 S_1^*, T_1^* D^{-1} T_1] = 0$, which are equivalent to (iii).

Note that $[S_1 S_1^*, T_1^* D^{-1} T_1] = 0$, since $(T_1 S_1)^\dagger T_1 S_1$ is selfadjoint. We show that (1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1).

Suppose (1) holds. If we multiply $(T_1 S_1)^\dagger T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1$ by $T_1 S_1$ from the left side, we obtain $T_1 = T_1 T_1^* D^{-1} T_1$. Furthermore, $[S_1 S_1^*, T_1^* D^{-1} T_1] = 0$, i.e. (3) holds.

Suppose (3) holds. Obviously, $T_1 S_1 S_1^* T_1^* D^{-1} T_1 = T_1 T_1^* D^{-1} T_1 S_1 S_1^* = T_1 S_1 S_1^*$, that is, the first equality of (2) holds. According to the fact that $(T_1 T_1^* + T_2 T_2^*) D^{-1} T_1 = T_1$ and the assumption $T_1 T_1^* D^{-1} T_1 = T_1$, we have $T_2^* D^{-1} T_1 = 0$. Consequently,

$$\text{Ran}(D^{-1} T_1) \subseteq \text{Ker}(T_2 T_2^*) = \text{Ker}(T_2^*),$$

which yields $T_2^* D^{-1} T_1 = 0$. Therefore, $T_1^* D^{-1} T_2 = 0$ which establishes the second equality of (2).

In order to prove (2) \Rightarrow (1), we multiply $T_1 S_1 S_1^* T_1^* D^{-1} T_1 = T_1 S_1 S_1^*$ by $(T_1 S_1)^\dagger$ from the left side. In view of $[S_1 S_1^*, T_1^* D^{-1} T_1] = 0$, we find

$$\begin{aligned} S_1^* T_1^* D^{-1} T_1 &= (T_1 S_1)^\dagger T_1 S_1 S_1^* \Rightarrow (T_1 S_1)^\dagger T_1 S_1 = S_1^* T_1^* D^{-1} T_1 (S_1^*)^{-1} \\ &\Leftrightarrow (T_1 S_1)^\dagger T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1. \end{aligned}$$

□

Now we are ready to derive Greville's result, which also gives an answer to a problem of [17] about the reverse order law for Moore-Penrose inverses of modular operators. The operators SS^\dagger and $T^\dagger T$ are orthogonal projections onto $\text{Ran}(S)$ and $\text{Ran}(T^\dagger) = \text{Ran}(T^*)$, respectively. These facts together with Theorems 2.1 and 2.2 lead us to the following result.

Corollary 2.3. *Suppose E, F and G are Hilbert C^* -modules and $S \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(F, G)$ and $TS \in \mathcal{L}(E, G)$ have closed ranges. Then following statements are equivalent:*

- (i) $(TS)^\dagger = S^\dagger T^\dagger$,
- (ii) $TS(TS)^\dagger = TSS^\dagger T^\dagger$ and $(TS)^\dagger TS = S^\dagger T^\dagger TS$,
- (iii) $SS^\dagger T^* TS = T^* TS$ and $TSS^* T^\dagger T = TSS^*$,
- (iv) $\text{Ran}(T^* TS) \subseteq \text{Ran}(S)$ and $\text{Ran}(SS^* T^*) \subseteq \text{Ran}(T^*)$.

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REFERENCES

- [1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, second ed., Springer, New York, 2003.
- [2] R. H. Bouldin, The pseudo-inverse of a product, *SIAM J. Appl. Math.* 25 (1973) 489-495.
- [3] D. S. Cvetković-Ilić and R. Harte, Reverse order laws in C^* -algebras, *Linear Algebra Appl.* **434** (2011), 1388-1394.
- [4] C. Y. Deng, Reverse order law for the group inverses, *J. Math. Anal. Appl.* **382** (2011) 663-671.
- [5] D. S. Djordjević and N. Č. Dinčić, Reverse order law for the Moore-Penrose inverse, *J. Math. Anal. Appl.* **361** (2010) 252-261.
- [6] M. Frank and K. Sharifi, Generalized inverses and polar decomposition of unbounded regular operators on Hilbert C^* -modules, *J. Operator Theory* **64** (2010), 377-386.
- [7] T. N. E. Greville, Note on the generalized inverse of a matrix product, *SIAM Rev.* **8** (1966) 518-521.
- [8] B. Guljaš, Unbounded operators on Hilbert C^* -modules over C^* -algebras of compact operators, *J. Operator Theory* **59**(2008), no. 1, 179-192.

- [9] S. Izumino, The product of operators with closed range and an extension of the reverse order law, *Tôhoku Math. J.* **34** (1982) 4352.
- [10] I. Kaplansky, Module over operator algebra, *Amer. J. Math.* **75** (1953), 839-858.
- [11] J. J. Koliha, D. S. Djordjević and D. S. Cvetković-Ilić, Moore-Penrose inverse in rings with involution, *Linear Algebra Appl.* **426** (2007), 371381.
- [12] E. C. Lance, *Hilbert C*-Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
- [13] D. Mosić and D. S. Djordjević, Reverse order law in C*-algebras, *Appl. Math. Comp.* **218** (2011) 3934-3941.
- [14] M. Moslehian, K. Sharifi, M. Forough and M. Chakoshi, Moore-Penrose inverse of Gram operator on Hilbert C*-modules, to appear in *Studia Math.*
- [15] W. L. Paschke, Inner product modules over B*-algebras, *Trans Amer. Math. Soc.* **182** (1973), 443-468.
- [16] K. Sharifi, Descriptions of partial isometries on Hilbert C*-modules, *Linear Algebra Appl.* **431** (2009), 883-887.
- [17] K. Sharifi, The product of operators with closed range in Hilbert C*-modules, *Linear Algebra Appl.* **435** (2011), 1122-1130.
- [18] K. Sharifi, Groetsch's representation of Moore-Penrose inverses and ill-posed problems in Hilbert C*-modules, *J. Math. Anal. Appl.* **365** (2010), 646-652.
- [19] Q. Xu and L. Sheng, Positive semi-definite matrices of adjointable operators on Hilbert C*-modules, *Linear Algebra Appl.* **428** (2008), 992-1000.

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